

# Maximum Weight Independent Set in $l$ Claw-Free Graphs in Polynomial Time

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*ad laudem Domini*

## Abstract

The Maximum Weight Independent Set (MWIS) problem is a well-known NP-hard problem. A popular way to study MWIS is to detect graph classes for which MWIS can be solved in polynomial time, with particular reference to hereditary graph classes, i.e., defined by a hereditary graph property or equivalently by forbidding one or more induced subgraphs.

For graphs  $G_1, G_2$ ,  $G_1 + G_2$  denotes the disjoint union of  $G_1$  and  $G_2$ , and for a constant  $l \geq 2$ ,  $lG$  denotes the disjoint union of  $l$  copies of  $G$ . A *claw* has vertices  $a, b, c, d$ , and edges  $ab, ac, ad$ . MWIS can be solved for claw-free graphs in polynomial time; the first two polynomial time algorithms were introduced in 1980 by [22, 29], then revisited by [24], and recently improved by [9, 10], and by [25, 26] with the best known time bound in [26]. Furthermore MWIS can be solved for the following extensions of claw-free graphs in polynomial time: fork-free graphs [19],  $K_2$ +claw-free graphs [20], and apple-free graphs [6, 7].

This manuscript shows that for any constant  $l$ , MWIS can be solved for  $l$ claw-free graphs in polynomial time. Our approach is based on Farber's approach showing that every  $2K_2$ -free graph has  $\mathcal{O}(n^2)$  maximal independent sets [11], which directly leads to a polynomial time algorithm for MWIS on  $2K_2$ -free graphs by dynamic programming.

Solving MWIS for  $l$ claw-free graphs in polynomial time extends known results for claw-free graphs, for  $lK_2$ -free graphs for any constant  $l$  [2, 12, 28, 30], for  $K_2$ +claw-free graphs, for  $2P_3$ -free graphs [21], and solves the open questions for  $2K_2 + P_3$ -free graphs and for  $P_3$ +claw-free graphs being two of the minimal graph classes, defined by forbidding one induced subgraph, for which the complexity of MWIS was an open problem.

# 1 Introduction

For any missing notation or reference let us refer to [5]. For a graph  $G$ , let  $V(G)$  ( $E(G)$ , respectively) denote its vertex set (edge set, respectively). For a subset  $U \subseteq V(G)$ , let  $N_G(U) = \{v \in V(G) \setminus U : v \text{ is adjacent to some } u \in U\}$  be the *neighborhood of  $U$  in  $G$* , and  $A_G(U) = V \setminus (U \cup N(U))$  be the *anti-neighborhood of  $U$  in  $G$* . If  $U = \{u_1, \dots, u_k\}$ , then let us simply write  $N_G(u_1, \dots, u_k)$  instead of  $N_G(U)$ , and  $A_G(u_1, \dots, u_k)$  instead of  $A_G(U)$ .

For  $U \subseteq V(G)$  let  $G[U]$  denote the subgraph of  $G$  induced by  $U$ . For a vertex  $v \in V(G)$  and for a subset  $U \subset V(G)$  (with  $v \notin U$ ), let us say that  $v$  *contacts*  $U$  if  $v$  is adjacent to some vertex of  $U$ , and  $v$  *dominates*  $U$  if  $v$  is adjacent to each vertex of  $U$ . A *component of  $G$*  is the vertex set of a maximal connected subgraph of  $G$ .

An *independent set* (or a *stable set*) of a graph  $G$  is a subset of pairwise nonadjacent vertices of  $G$ . An independent set of  $G$  is *maximal* if it is not properly contained in any other independent set of  $G$ .

For a given graph  $H$ , a graph  $G$  is  *$H$ -free* if none of its induced subgraphs is isomorphic to  $H$ ; in particular,  $H$  is called a *forbidden induced subgraph of  $G$* . Given two graphs  $G$  and  $F$ ,  $G + F$  denotes the disjoint union of  $G$  and  $F$ ; in particular,  $2G = G + G$  and in general, for  $l \geq 2$ ,  $lG$  denotes the disjoint union of  $l$  copies of  $G$ .

The following specific graphs are mentioned later. A *chordless path*  $P_k$  has vertices  $v_1, v_2, \dots, v_k$  and edges  $v_j v_{j+1}$  for  $1 \leq j < k$ . A *chordless cycle*  $C_k$ ,  $k \geq 4$ , has vertices  $v_1, v_2, \dots, v_k$  and edges  $v_j v_{j+1}$  for  $1 \leq j < k$  and  $v_k v_1$ . A  $K_n$  is a complete graph of  $n$  vertices. A  $K_{1,n}$  is a complete bipartite graph whose sides respectively have one vertex, called the *center* of  $K_{1,n}$ , and  $n$  vertices, called the *leaves* of  $K_{1,n}$  (if  $n = 1$  then there are two trivial centers).  $K_{1,3}$  is also called *claw*.

A *fork* (sometimes called *chair*) has vertices  $a, b, c, d, e$ , and edges  $ab, ac, ad, de$  (thus, a fork contains a claw as an induced subgraph). An *apple* is formed by a  $C_k$ ,  $k \geq 4$ , plus one vertex adjacent to exactly one vertex of the  $C_k$ .

For indices  $i, j, k \geq 0$ , let  $S_{i,j,k}$  denote the graph with vertices  $u, x_1, \dots, x_i, y_1, \dots, y_j, z_1, \dots, z_k$  such that the subgraph induced by  $u, x_1, \dots, x_i$  forms a  $P_{i+1}$  ( $u, x_1, \dots, x_i$ ), the subgraph induced by  $u, y_1, \dots, y_j$  forms a  $P_{j+1}$  ( $u, y_1, \dots, y_j$ ), and the subgraph induced by  $u, z_1, \dots, z_k$  forms a  $P_{k+1}$  ( $u, z_1, \dots, z_k$ ), and there are no other edges in  $S_{i,j,k}$ . Thus, *claw* is  $S_{1,1,1}$ , and  $P_k$  is isomorphic to e.g.  $S_{0,0,k-1}$ .

Let  $G$  be a given graph and let  $w$  be a weight function on  $V(G)$ . For an independent set  $I$ , its weight is  $w(I) := \sum_{v \in I} w(v)$ . Let  $\alpha_w(G) := \max\{w(I) : I \text{ independent in } G\}$  denote the maximum weight of any independent set of  $G$ .

The *Maximum Weight Independent Set (MWIS)* problem asks for an independent set of  $G$  of maximum weight.

If all vertices  $v$  have the same weight  $w(v) = 1$ ,  $\alpha_w(G) = \alpha(G)$  and MWIS is called the *MIS* problem.

MWIS is NP-hard [16] and remains NP-hard under various restrictions, such as for

triangle-free graphs [27] and more generally for graphs without chordless cycle of given length [23], for cubic graphs [15] and more generally for  $k$ -regular graphs [13], and for planar graphs [14].

It can be solved in polynomial time for various graph classes, such as for  $P_4$ -free graphs [8] and more generally perfect graphs [17], for claw-free graphs [9, 22, 24, 25, 29] and more generally fork-free graphs [3, 19] and apple-free graphs [6, 7], for  $2K_2$ -free graphs [11] and more generally  $lK_2$ -free graphs for any constant  $l$  (by combining an algorithm generating all maximal independent sets of a graph [30] and a polynomial upper bound on the number of maximal independent sets in  $lK_2$ -free graphs [2, 12, 28]),  $K_2$ +claw-free graphs [20], and  $2P_3$ -free graphs [21]. Furthermore MWIS can be solved in polynomial time for  $P_5$ -free graphs as recently proved in [18].

The first two polynomial time algorithms for MWIS on claw-free graphs were introduced in 1980 by Minty [22] and independently by Sbihi [29], then revisited by Nakamura and Tamura [24], and recently improved by Faenza, Oriolo, and Stauffer [9, 10], and by Nobili and Sassano [25, 26] with the best known time bound in [26].

**Theorem 1 [26]** *For claw-free graphs, the MWIS problem can be solved in time  $O(n^2 \log n)$ .  $\square$*

Obviously, for every graph  $G$  the following holds:

$$\alpha_w(G) = \max\{w(v) + \alpha_w(G[A(v)]) : v \in V\}$$

Thus, for any graph  $G$ , MWIS can be reduced to the same problem for the anti-neighborhoods of all vertices of  $G$ . Then we have:

**Proposition 1** *For any graph  $F$ , if  $M(W)IS$  can be solved for  $F$ -free graphs in polynomial time then  $M(W)IS$  can be solved for  $K_1 + F$ -free graphs in polynomial time.  $\square$*

Let us report the following result due to Alekseev [1, 4]. Let us say that a graph is of *type  $T$*  if it is graph  $S_{i,j,k}$  for some indices  $i, j, k$ .

**Theorem 2 [1]** *Let  $\mathcal{X}$  be a class of graphs defined by a finite set  $M$  of forbidden induced subgraphs. If  $M$  does not contain any graph every connected component of which is of type  $T$ , then the  $M(W)IS$  problem is NP-hard for the class  $\mathcal{X}$ .*

Alekseev's result implies that  $M(W)IS$  is NP-hard for  $K_{1,4}$ -free graphs — the fact that  $M(W)IS$  is NP-hard for  $K_{1,4}$ -free graphs is already mentioned in [22].

Unless  $P = NP$ , Alekseev's result implies that for any graph  $F$ , if  $M(W)IS$  is polynomial time solvable for  $F$ -free graphs, then each connected component of  $F$  is of type  $T$ . By Proposition 1, for any graph  $F$ , if  $M(W)IS$  can be solved in polynomial time for  $F$ -free graphs then for any constant  $l$ ,  $M(W)IS$  can be solved in polynomial time for

$lK_1 + F$ -free graphs. It follows that, since for any constant  $l$ , M(W)IS can be solved in polynomial time for  $lK_2$ -free graphs [2, 12, 28, 30], for fork-free graphs [3, 19], for  $K_2$ +claw-free graphs [20], for  $2P_3$ -free graphs [21], and for  $P_5$ -free graphs [18], the minimal graphs  $F$  of type  $T$  for which the complexity of M(W)IS for  $F$ -free graphs was open are:  $P_6$ ,  $S_{1,1,3}$ ,  $S_{1,2,2}$ ,  $K_2 + P_4$ ,  $2K_2 + P_3$ ,  $P_3$ +claw, and thus, the minimal graph classes, defined by forbidding one induced subgraph, for which the complexity of M(W)IS was open are:

$P_6$ -free graphs,  $S_{1,1,3}$ -free graphs,  $S_{1,2,2}$ -free graphs,  $K_2 + P_4$ -free graphs,  $2K_2 + P_3$ -free graphs,  $P_3$ +claw-free graphs.

In this manuscript, we show that for any constant  $l$ , MWIS can be solved for  $l$ claw-free graphs in polynomial time. This extends the known results for MWIS on claw-free graphs,  $lK_2$ -free graphs for any constant  $l$ ,  $K_2$ +claw-free graphs,  $2P_3$ -free graphs, and solves the open question for MWIS on  $2K_2 + P_3$ -free graphs and on  $P_3$ +claw-free graphs.

Our approach is based on Farber's approach showing that every  $2K_2$ -free graph has  $\mathcal{O}(n^2)$  maximal independent sets [11] (reported in Section 2), which directly leads to a polynomial time algorithm to solve MWIS for  $2K_2$ -free graphs by dynamic programming.

## 2 Maximal Independent Sets in $2K_2$ -Free Graphs

In this section let us refer to Algorithm  $\mathcal{A}$  (subsequently called Algorithm Alpha) from [20] which formalizes the aforementioned approach by Farber [11]; our subsequent approach for MWIS on  $l$ claw-free graphs is based on this algorithm.

For a  $2K_2$ -free input graph  $G$ , Algorithm Alpha produces a family  $\mathcal{S}$  of independent sets of  $G$ , which can be computed in time  $\mathcal{O}(n^3)$  and which contains  $\mathcal{O}(n^2)$  members such that each maximal independent set of  $G$  is contained in some member of  $\mathcal{S}$ .

For a graph  $G = (V, E)$  with  $|V| = n$ , a *vertex ordering*  $(v_1, v_2, \dots, v_n)$  of  $G$  is a total ordering of the vertex set  $V$  of  $G$ . For such a vertex ordering  $(v_1, v_2, \dots, v_n)$  of  $G$ , let  $G_i := G[\{v_1, v_2, \dots, v_i\}]$  denote the subgraph of  $G$  induced by the first  $i$  vertices,  $i \geq 1$ .

Given a vertex ordering  $(v_1, v_2, \dots, v_n)$ , at each loop  $i$ ,  $1 \leq i \leq n$ , Algorithm Alpha provides a family  $\mathcal{S}_i$  of subsets of  $\{v_1, v_2, \dots, v_i\}$  (by modifying  $\mathcal{S}$  at loop  $i$  by extending some of its members or by adding new members) such that each maximal independent set of  $G_i$  is contained in some member of  $\mathcal{S}_i$ , and finally returns the family  $\mathcal{S}_n = \mathcal{S}$ .

### Algorithm Alpha

**Input:** A  $2K_2$ -free graph  $G$  and a vertex ordering  $(v_1, v_2, \dots, v_n)$  of  $G$ .

**Output:** A family  $\mathcal{S}$  of subsets of  $V(G)$ .

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 $\mathcal{S} := \{\emptyset\};$ 
For  $i =: 1$  to  $n$  do
begin

```

1. [Extension of some members of  $\mathcal{S}$ ]  
**For each**  $H \in \mathcal{S}$  **do**  
    **If**  $H \cup \{v_i\}$  is an independent set **then**  $H := H \cup \{v_i\}$ .
  2. [Addition of new members to  $\mathcal{S}$ ]  
**For each**  $K_2$  of  $G_i$  containing  $v_i$  (i.e., for each edge  $uv_i$  of  $G_i$ ) **do**  
     $H := \{v_i\} \cup A_{G_i}(u, v_i)$ ;  
     $\mathcal{S} := \mathcal{S} \cup \{H\}$ .
- end.**

Then the MWIS problem can be solved for  $2K_2$ -free graphs by the following algorithm.

**Algorithm  $2K_2$ -Free-MWIS**

**Input:** A  $2K_2$ -free graph  $G$ .

**Output:** A maximum weight independent set of  $G$ .

- (1) Execute Algorithm Alpha for  $G$ . Let  $\mathcal{S}$  be the resulting family of subsets of  $V(G)$ .
- (2) For each  $H \in \mathcal{S}$ , compute a maximum weight independent set of  $G[H]$  (note that each  $H \in \mathcal{S}$  is an independent set since  $G$  is  $2K_2$ -free). Then choose a best solution, i.e., one of maximum weight.

Then one obtains the following result.

**Theorem 3 [11]** *For  $2K_2$ -free graphs, the MWIS problem can be solved in time  $\mathcal{O}(n^4)$  by Algorithm  $2K_2$ -Free-MWIS.*  $\square$

## 3 Maximal Independent Sets in Claw+Claw-Free Graphs

### 3.1 A Basic Lemma

First let us introduce a preparatory result. For each  $k \in \{1, \dots, 14\}$ , let  $L_k$  be the graph drawn in the subsequent figure. Note that each  $L_k$  contains an induced claw. For each  $k \in \{1, \dots, 14\}$ , let  $W(L_k)$  denote the set of white vertices of  $L_k$ , let  $B(L_k)$  denote the set of black vertices of  $L_k$ , and let  $top(L_k)$  denote the (white) vertex at the top of  $L_k$ .

**Lemma 1** *For a graph  $G$ , assume that  $v \in V(G)$  is a vertex such that  $v$  is contained in an induced claw of  $G$  and  $G[V(G) \setminus \{v\}]$  is claw-free. Then for each maximal independent set  $I$  of  $G$  with  $v \in I$ , there is a  $k \in \{1, \dots, 14\}$  such that  $I \subseteq W(L_k) \cup A_G(L_k)$  for an induced subgraph  $L_k$  of  $G$  with  $v = top(L_k)$ .*

**Proof.** Let  $K$  be a claw in  $G$  with, say,  $V(K) = \{v, a, b, c\}$ . Let  $I$  be a maximal independent set of  $G$  containing  $v$ , and let  $I' := I \setminus \{v\}$ . Then for  $H := V(G) \setminus \{v\}$ ,  $I'$

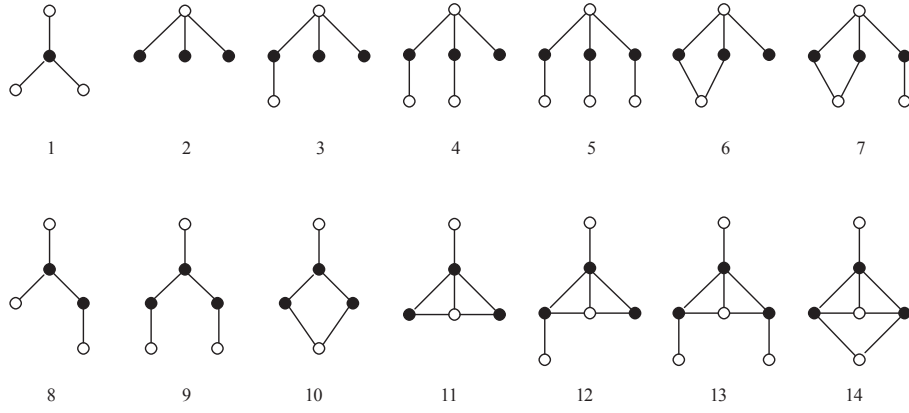


Figure 1: Graphs  $L_k$  for  $k = 1, \dots, 14$

is a maximal independent set of  $G[H \setminus N(v)]$ . Let us distinguish between the following cases.

**Case 1**  $G[H]$  is connected.

By assumption,  $v$  is contained in an induced claw of  $G$ . Let us distinguish between two subcases.

**Case 1.1**  $v$  is the center of  $K$ .

Since  $G[H]$  is claw-free, each of  $a, b, c$  has at most two neighbors in  $I'$ .

**Case 1.1.1** If a vertex of  $a, b, c$ , say  $a$ , has two neighbors in  $I'$ , say  $s_1, s_2$  then  $I \subseteq W(L_1) \cup A_G(L_1)$  with  $V(L_1) = \{v, a, s_1, s_2\}$ ,  $W(L_1) = \{v, s_1, s_2\}$ , and  $v = \text{top}(L_1)$ .

**Case 1.1.2** If none of  $a, b, c$  has a neighbor in  $I'$  then  $I \subseteq W(L_2) \cup A_G(L_2)$  with  $V(L_2) = \{v, a, b, c\}$  and  $v = \text{top}(L_2)$ .

**Case 1.1.3** Now assume that Cases 1.1.1 and 1.1.2 are excluded. This means that one of  $a, b, c$ , say without loss of generality  $a$ , has exactly one neighbor in  $I'$  and  $b$  and  $c$  have at most one neighbor in  $I'$ . Let  $as_1 \in E$  for  $s_1 \in I'$ . Note that not both of  $b$  and  $c$  are adjacent to  $s_1$  since  $H$  is claw-free, and in general,  $a, b$  and  $c$  do not have any common neighbor in  $I'$ .

If  $N(b) \cap I' = N(c) \cap I' = \emptyset$  then we have  $I \subseteq W(L_3) \cup A_G(L_3)$  with  $V(L_3) = \{v, a, b, c, s_1\}$  and  $v = \text{top}(L_3)$ .

If  $b$  has exactly one neighbor in  $I'$ , say  $s_2$ , and  $N(c) \cap I' = \emptyset$  then if  $s_1 \neq s_2$ , we have  $I \subseteq W(L_4) \cup A_G(L_4)$  with  $V(L_4) = \{v, a, b, c, s_1, s_2\}$  and  $v = \text{top}(L_4)$ , and if  $s_1 = s_2$ , we have  $I \subseteq W(L_6) \cup A_G(L_6)$  with  $V(L_6) = \{v, a, b, c, s_1\}$  and  $v = \text{top}(L_6)$ , and similarly for the case when  $c$  has exactly one neighbor in  $I'$ , and  $N(b) \cap I' = \emptyset$ .

Finally, assume that both  $b$  and  $c$  have a neighbor in  $I'$ , i.e., there are  $s_2, s_3 \in I'$  with  $bs_2 \in E$  and  $cs_3 \in E$ .

If  $s_1, s_2, s_3$  are pairwise distinct then we have  $I \subseteq W(L_5) \cup A_G(L_5)$  with  $V(L_5) = \{v, a, b, c, s_1, s_2, s_3\}$  and  $v = \text{top}(L_5)$ .

Now assume that  $|\{s_1, s_2, s_3\}| = 2$  (recall that  $|\{s_1, s_2, s_3\}| = 1$  is impossible). Without loss of generality, let  $s_1 = s_2$ . Then we have  $I \subseteq W(L_7) \cup A_G(L_7)$  with  $V(L_7) = \{v, a, b, c, s_1, s_3\}$  and  $v = \text{top}(L_7)$ .

**Case 1.2**  $v$  is a leaf of  $K$ .

Without loss of generality, let  $b$  be the center of  $K$ . Since  $G[H]$  is claw-free,  $b$  has at most two neighbors in  $I'$ , and if  $a \notin I'$  ( $c \notin I'$ , respectively), the same holds for  $a$  ( $c$ , respectively).

The following subcases are exhaustive by symmetry.

**Case 1.2.1** If  $a, c \in I'$  then  $I \subseteq W(L_1) \cup A_G(L_1)$  with  $V(L_1) = \{v, a, b, c\}$  and  $v = \text{top}(L_1)$ .

**Case 1.2.2** If exactly one of  $a, c$  is in  $I'$ , say without loss of generality,  $a \in I'$  and  $c \notin I'$  (and more generally, only one of the neighbors of  $b$  is in  $I'$  - otherwise we

have Case 1.2.1) then  $c$  has a neighbor in  $I'$ , say  $s$ , since  $I'$  is a maximal independent set of  $G[H \setminus N(v)]$ . Then clearly,  $s$  is nonadjacent to  $a$  and  $v$  and is nonadjacent to  $b$  (otherwise  $b$  would have two neighbors in  $I'$ ). Then  $I \subseteq W(L_8) \cup A_G(L_8)$  with  $V(L_8) = \{v, a, b, c, s\}$  and  $v = \text{top}(L_8)$ .

**Case 1.2.3** Now assume that Cases 1.2.1 and 1.2.2 are excluded. Thus,  $a, c \notin I'$ . Then both  $a$  and  $c$  must have a neighbor in  $I'$  since  $I'$  is a maximal independent set of  $G[H \setminus N(v)]$ .

If no neighbor of  $a$  or  $c$  in  $I'$  is adjacent to  $b$  then both  $a$  and  $c$  have exactly one neighbor in  $I'$ , else a claw in  $G[H]$  would arise involving  $b$ . Let  $s_1, s_2 \in I'$  with  $as_1 \in E$ ,  $cs_2 \in E$ .

If  $s_1 \neq s_2$  then  $I \subseteq W(L_9) \cup A_G(L_9)$  with  $V(L_9) = \{v, a, b, c, s_1, s_2\}$  and  $v = \text{top}(L_9)$ .

If  $s_1 = s_2$  then  $I \subseteq W(L_{10}) \cup A_G(L_{10})$  with  $V(L_{10}) = \{v, a, b, c, s_1\}$  and  $v = \text{top}(L_{10})$ .

Now assume that, without loss of generality, a neighbor  $s \in I'$  of  $a$  is adjacent to  $b$ . We claim:

- (i)  $s$  is adjacent to  $c$ , since otherwise Case 1.2.2 holds with  $s$  instead of  $a$ ;
- (ii)  $a$  and  $c$  have at most one more neighbor in  $I'$ , and such a neighbor is non-adjacent to  $b$ , since otherwise Case 1.2.1 holds (i.e.,  $b$  has two neighbors in  $I'$ ).

If neither  $a$  nor  $c$  have another neighbor in  $I'$  then  $I \subseteq W(L_{11}) \cup A_G(L_{11})$  with  $V(L_{11}) = \{v, a, b, c, s\}$  and  $v = \text{top}(L_{11})$ .

If there is  $s_1 \in I'$  with  $s_1 \neq s$ ,  $as_1 \in E$  and the only neighbor of  $c$  in  $I'$  is  $s$  then  $I \subseteq W(L_{12}) \cup A_G(L_{12})$  with  $V(L_{12}) = \{v, a, b, c, s, s_1\}$  and  $v = \text{top}(L_{12})$ , and similarly if  $c$  has two neighbors  $s, s_1 \in I'$  and  $a$  has only neighbor  $s \in I'$ .

Finally, if  $a$  and  $c$  have another neighbor in  $I'$ , say  $s_1, s_2 \in I'$ ,  $s \neq s_1, s \neq s_2$  with  $as_1 \in E$  and  $cs_2 \in E$  then we have:

If  $s_1 \neq s_2$  then  $I \subseteq W(L_{13}) \cup A_G(L_{13})$  with  $V(L_{13}) = \{v, a, b, c, s, s_1, s_2\}$  and  $v = \text{top}(L_{13})$ , and if  $s_1 = s_2$  then  $I \subseteq W(L_{14}) \cup A_G(L_{14})$  with  $V(L_{14}) = \{v, a, b, c, s, s_1\}$  and  $v = \text{top}(L_{14})$ .

**Case 2:**  $G[H]$  is not connected.

This case can be treated similarly as Case 1 in which  $G[H]$  is connected. If  $G$  is not connected then we can solve MWIS separately for each component of  $G$ . If  $G$  is connected and  $v$  is the leaf of a claw then obviously,  $G[H]$  is connected. Thus, we can assume that  $v$  is the center of a claw  $K$ , and we can follow the arguments of Case 1. For brevity let us omit the proof, which can be split into the subcases in which  $G[H]$  has two or three components. Finally we have  $I \subseteq W(L_k) \cup A_G(L_k)$ , for some  $k \in \{1, \dots, 7\}$ , with  $v = \text{top}(L_k)$ .  $\square$

### 3.2 MWIS for Claw+Claw-Free Graphs

Now we show that for claw+claw-free graphs, MWIS can be solved in time  $\mathcal{O}(n^{10})$ . For this, we need the following notion:



**Definition 1** Let  $G$  be a graph and let  $\mathcal{S}$  be a family of subsets of  $V(G)$ . Then  $\mathcal{S}$  is a good claw-free family of  $G$  if the following holds:

- (i) Each member of  $\mathcal{S}$  induces a claw-free subgraph in  $G$ .
- (ii) Each maximal independent set of  $G$  is contained in some member of  $\mathcal{S}$ .
- (iii)  $\mathcal{S}$  contains polynomially many members and can be computed in polynomial time.

The basic step is the subsequent Algorithm Gamma(2) (based on the corresponding Algorithm Alpha of Section 2) which, for any claw+claw-free (i.e., 2claw-free) input graph  $G$ , computes a good claw-free family  $\mathcal{S}$  of  $G$ . The approach is based on Farber's idea for MWIS on  $2K_2$ -free graphs described in Algorithm Alpha of Section 2.

**Algorithm Gamma(2)**

**Input:** A claw+claw-free graph  $G$  and a vertex-ordering  $(v_1, v_2, \dots, v_n)$  of  $G$ .

**Output:** A good claw-free family  $\mathcal{S}$  of  $G$ .

$\mathcal{S} := \{\emptyset\};$

**For**  $i =: 1$  **to**  $n$  **do**

**begin**

1. [Extension of some members of  $\mathcal{S}$ ]

**For each**  $H \in \mathcal{S}$  **do**

**If**  $G[H \cup \{v_i\}]$  is claw-free **then**  $H := H \cup \{v_i\}$ .

2. [Addition of new members to  $\mathcal{S}$ ]

**For each** induced  $L_k$  of  $G_i$ ,  $k \in \{1, \dots, 14\}$ , with  $v_i = \text{top}(L_k)$  **do**

        Compute a good claw-free family, say  $\mathcal{F}$ , of  $G[A_{G_i}(L_k)]$ .

**For each**  $F \in \mathcal{F}$ , set  $\mathcal{S} := \mathcal{S} \cup \{W(L_k) \cup F\}$ .

**end.**

**Proposition 2** Step 2 of Algorithm Gamma(2) is well defined, i.e.,  $G[A_{G_i}(L_k)]$  is claw-free and has a good claw-free family (formed by one member, namely,  $A_{G_i}(L_k)$ ) which can be computed in constant time.

**Proof.** Subgraph  $G[A_{G_i}(L_k)]$  is claw-free since  $G$  is assumed to be claw+claw-free, each  $L_k$  contains an induced claw and  $A_{G_i}(L_k)$  is defined as the anti-neighborhood of  $V(L_k)$ . Then the subgraph  $G[A_{G_i}(L_k)]$  has a good claw-free family (formed by one member, namely,  $A_{G_i}(L_k)$ ) which can be computed in constant time.  $\square$

For proving the correctness and the time bound of Algorithm Gamma(2), we need the following lemmas.

**Lemma 2** Let  $G$  be a claw+claw-free graph and let  $\mathcal{S}$  be the result of Algorithm Gamma(2). Then we have:

- (i) Each member of  $\mathcal{S}$  induces a claw-free subgraph of  $G$ .

(ii) Each maximal independent set of  $G$  is contained in some member of  $\mathcal{S}$ .

**Proof.** (i): Each member of  $\mathcal{S}$  is created either in the initialization step as the empty set or in Step 1 or Step 2 of some loop. Clearly, each member  $H \cup \{v_i\}$  created in Step 1 induces a claw-free subgraph in  $G$  since each member of  $\mathcal{S}$  is extended in Step 1 only if the extension preserves its claw-freeness. According to Step 2 and to Proposition 2, each member of  $\mathcal{S}$  created in Step 2 is the disjoint union of a vertex subset of a claw-free subgraph, namely  $W(L_k)$ , and of a claw-free subgraph representing its anti-neighborhood  $A_{G_i}(L_k)$ , namely a member of a good claw-free family. Therefore, each member of  $\mathcal{S}$  created in Step 2 induces a claw-free graph. This completes the proof of statement (i).

(ii): By  $\mathcal{S}_i$ , let us denote the family  $\mathcal{S}$  resulting by the  $i$ -th loop of Algorithm Gamma(2). Let us show that for all  $i \in \{1, \dots, n\}$ , each maximal independent set of  $G_i$  is contained in a member  $H$  of  $\mathcal{S}_i$ . The proof is done by induction. For  $i = 1$ , the statement is trivial. Then let us assume that the statement holds for  $i - 1$  and prove that it holds for  $i$ .

Let  $I$  be a maximal independent set of  $G_i$ .

If  $v_i \notin I$ , then by the induction assumption,  $I$  is contained in some member of  $\mathcal{S}_{i-1}$ , and thus of  $\mathcal{S}_i$ , since each member of  $\mathcal{S}_{i-1}$  is a (not necessarily proper) subset of a member of  $\mathcal{S}_i$ .

If  $v_i \in I$ , then let us consider the following argument. By the induction assumption, let  $H \in \mathcal{S}_{i-1}$  with  $I \setminus \{v_i\} \subseteq H$ . Note that for all  $j$ ,  $1 \leq j \leq n$ , each member of  $\mathcal{S}_j$  induces a claw-free graph, as one can easily verify by an argument similar to the proof of statement (i). Thus,  $G[H]$  is claw-free.

Then let us consider the following two cases which are exhaustive by definition of Algorithm Gamma(2).

**Case 1:**  $G[H \cup \{v_i\}]$  is claw-free.

Then  $I$  is contained in the set  $H \cup \{v_i\}$ , which is a member of  $\mathcal{S}_i$  since it is generated by Step 1 of the algorithm at loop  $i$ .

**Case 2:**  $G[H \cup \{v_i\}]$  is not claw-free.

Then by Lemma 1, since  $G[H]$  is claw-free, there is a  $k \in \{1, \dots, 14\}$  such that  $I \subseteq W(L_k) \cup A_{G_i}(L_k)$  for an induced subgraph  $L_k$  of  $G_i$  with  $v_i = \text{top}(L_k)$ , and  $W(L_k) \cup A_{G_i}(L_k)$  is contained in  $\mathcal{S}_i$  since it is generated by Step 2 of Algorithm Gamma(2) at loop  $i$ .  $\square$

**Lemma 3** *The family  $\mathcal{S}$  produced by Algorithm Gamma(2) contains  $\mathcal{O}(n^7)$  members and can be computed in  $\mathcal{O}(n^9)$  time, which is also the time bound of Algorithm Gamma(2).*

**Proof.** The members of  $\mathcal{S}$  are created either in the initialization step or in Step 2 of all the loops of Algorithm Gamma(2) and then are possibly (iteratively) extended in Step 1 of Algorithm Gamma(2).

Concerning the member created in the initialization step, i.e., the empty set: This member is created in constant time and is possibly (iteratively) extended by Step 1 of each loop in  $\mathcal{O}(n)$  time (and the number of loops is  $n$ ). Then this member can be computed in  $\mathcal{O}(n^2)$  time.

Concerning the members created in Step 2 of all the loops: Such members are created with respect to all induced  $L_k$ ,  $1 \leq k \leq 14$  (the maximum number of vertices in any  $L_k$  is 7), of  $G_i$ , i.e., with respect to a family of  $\mathcal{O}(n^7)$  subsets of  $G_i$  (in fact the algorithm produces the anti-neighborhoods of all  $L_k$  for  $k \in \{1, \dots, 14\}$  of  $G_i$  just once since at loop  $i$  all such  $L_k$  contain  $v_i$ ). Then for the respective anti-neighborhood, namely  $A_{G_i}(L_k)$ , of each such subset the algorithm computes a good claw-free family. By Proposition 2,  $G[A_{G_i}(L_k)]$  is claw-free and has a good claw-free family (which contains one member and can be computed in constant time). Therefore the cardinality of the family of such members is  $\mathcal{O}(n^7)$  and all such members can be created in  $\mathcal{O}(n^7)$  time (since each such member can be created in Step 2 in constant time). Then such members are possibly (iteratively) extended in Step 1 in  $\mathcal{O}(n)$  time (and the number of loops is  $n$ ). Then such members can be computed in  $\mathcal{O}(n^9)$  time.

Therefore,  $S$  contains  $\mathcal{O}(n^7)$  members and can be computed in  $\mathcal{O}(n^9)$  time, which is also the time bound of Algorithm Gamma(2).  $\square$

Note that Lemmas 2 and 3 directly imply the following.

**Corollary 1** *Every claw+claw-free graph has a good claw-free family which can be computed by Algorithm Gamma(2).*  $\square$

Then the MWIS problem can be solved for claw+claw-free graphs by the following algorithm.

**Algorithm MWIS(2)**

**Input:** A claw+claw-free graph  $G$ .

**Output:** A maximum weight independent set of  $G$ .

- (1) Execute Algorithm Gamma(2) for  $G$ . Let  $\mathcal{S}$  be the resulting family of subsets of  $V(G)$ .
- (2) For each  $H \in \mathcal{S}$ , compute a maximum weight independent set of  $G[H]$ . Then choose a best solution, i.e., one of maximum weight.

**Theorem 4** *Algorithm MWIS(2) is correct and can be done in  $\mathcal{O}(n^{10})$  time.*

**Proof.** *Correctness:* By Lemma 2 (ii), Algorithm MWIS(2) is correct.

*Time bound:* By Lemma 3, step (1) can be executed in  $\mathcal{O}(n^9)$  time. By Lemma 3, the family  $\mathcal{S}$  contains  $\mathcal{O}(n^7)$  members. Then, by Lemma 2 (i) and Theorem 1, step (2) can be executed in  $\mathcal{O}(n^{10})$  time. Thus, Algorithm MWIS(2) can be executed in time  $\mathcal{O}(n^{10})$ .  $\square$

Then one obtains the following result.

**Corollary 2** *For claw+claw-free graphs, the MWIS problem can be solved in time  $\mathcal{O}(n^{10})$  by Algorithm MWIS(2).*  $\square$

## 4 MWIS for $l$ Claw-Free Graphs

In this section we show that for any fixed  $l \geq 2$ , MWIS for  $l$ claw-free graphs can be solved in polynomial time. For this, we first describe the subsequent Algorithm Gamma( $l$ ), which for any  $l$ claw-free input graph  $G$  computes a good claw-free family  $\mathcal{S}$  of  $G$ . The approach recursively uses Algorithm Gamma( $l-1$ ) for Algorithm Gamma( $l$ ), starting with Algorithm Gamma(2) of subsection 3.2.

**Algorithm Gamma( $l$ )**

**Input:** An  $l$ claw-free graph  $G$  and a vertex-ordering  $(v_1, v_2, \dots, v_n)$  of  $G$ .

**Output:** A good claw-free family  $\mathcal{S}$  of  $G$ .

$\mathcal{S} := \{\emptyset\};$

**For**  $i =: 1$  **to**  $n$  **do**

**begin**

1. [Extension of some members of  $\mathcal{S}$ ]

**For each**  $H \in \mathcal{S}$  **do**

**If**  $G[H \cup \{v_i\}]$  is claw-free **then**  $H := H \cup \{v_i\}$ .

2. [Addition of new members to  $\mathcal{S}$ ]

**For each** induced  $L_k$  of  $G_i$ ,  $k \in \{1, \dots, 14\}$ , with  $v_i = \text{top}(L_k)$  **do**

Compute a good claw-free family, say  $\mathcal{F}$ , of  $G[A_{G_i}(L_k)]$  by Algorithm Gamma( $l-1$ ).

**For each**  $F \in \mathcal{F}$ , set  $\mathcal{S} := \mathcal{S} \cup \{W(L_k) \cup F\}$ .

**end.**

**Assumption 1.** To prove the subsequent Proposition 3, Lemmas 4 and 5, and Corollary 3, we need to consider them as a *unique* result, in order to give a proof by induction on  $l$ . For  $l = 2$ , the proof of Proposition 3, of Lemmas 4 and 5, and of Corollary 3 is respectively that of Proposition 2, of Lemmas 2 and 3, and of Corollary 1.

Then let us assume that the subsequent Proposition 3, Lemmas 4 and 5, and Corollary 3 hold for  $l-1$  and let us show that they hold for  $l$ .

**Proposition 3** *Step 2 of Algorithm Gamma( $l$ ) is well defined, i.e.,  $G[A_{G_i}(L_k)]$  is  $(l-1)$ claw-free and has a good claw-free family which can be computed by Algorithm Gamma( $l-1$ ).*

**Proof.** Subgraph  $G[A_{G_i}(L_k)]$  is  $(l-1)$ claw-free since  $G$  is  $l$ claw-free and since  $A_{G_i}(L_k)$  is defined as the anti-neighborhood of  $L_k$  containing an induced claw. Then by Assumption 1, i.e., by Corollary 3 with respect to  $l-1$ , subgraph  $G[A_{G_i}(L_k)]$  has a good claw-free family which can be computed by Algorithm Gamma( $l-1$ ).  $\square$

For proving the correctness and the time bound of Algorithm Gamma( $l$ ), we need the following lemmas.

**Lemma 4** *Let  $G$  be an  $l$ claw-free graph and let  $\mathcal{S}$  be the result of Algorithm Gamma( $l$ ). Then we have:*

- (i) *Each member of  $\mathcal{S}$  induces a claw-free subgraph of  $G$ .*
- (ii) *Each maximal independent set of  $G$  is contained in some member of  $\mathcal{S}$ .*

**Proof.** According to Assumption 1, the proof is similar to that of Lemma 2, with Proposition 3 instead of Proposition 2 and with Algorithm Gamma( $l$ ) instead of Algorithm Gamma(2).  $\square$

**Lemma 5** *The family  $\mathcal{S}$  produced by Algorithm Gamma( $l$ ) contains polynomially many members and can be computed in polynomial time, which is also the time bound of Algorithm Gamma( $l$ ).*

**Proof.** The members of  $\mathcal{S}$  are created either in the initialization step or in Step 2 of all the loops of Algorithm Gamma( $l$ ) and then are possibly (iteratively) extended in Step 1 of Algorithm Gamma( $l$ ).

Concerning the member created in the initialization step, i.e., the empty set, this member is created in constant time and is possibly (iteratively) extended by Step 1 of each loop in  $\mathcal{O}(n)$  time (the number of loops is  $n$ ). Then this member can be computed in  $\mathcal{O}(n^2)$  time.

Concerning the members created in Steps 2 of all the loops, such members are created with respect to all induced  $L_k$  of  $G$ , i.e., with respect to a family of  $\mathcal{O}(n^7)$  subsets of  $G$  (in fact, the algorithm produces the anti-neighborhoods of all  $L_k$  for  $k \in \{1, \dots, 14\}$  of  $G_i$  just once since at loop  $i$  all such  $L_k$  contain  $v_i$  as their top vertex). Then for the respective anti-neighborhood, namely  $A_{G_i}(L_k)$ , of each such subset the algorithm computes a good claw-free family. By Proposition 3,  $G[A_{G_i}(L_k)]$  is  $(l-1)$ claw-free and has a good claw-free family. Therefore the cardinality of the family of such members is bounded by a polynomial and all such members can be created in polynomial time (since each such member can be created in Step 2 in polynomial time). Then such members are possibly (iteratively) extended in Step 1 in  $\mathcal{O}(n)$  time (the number of loops is  $n$ ). Thus, such members can be computed in polynomial time.

Therefore,  $\mathcal{S}$  can be computed in polynomial time, which is also the time bound of Algorithm Gamma( $l$ ).  $\square$

Note that Lemmas 4 and 5 directly imply the following.

**Corollary 3** *For any fixed  $l$ , each  $l$ claw-free graph has a good claw-free family which can be computed via Algorithm Gamma( $l$ ).*  $\square$

Then for  $l$ claw-free graphs, the MWIS problem can be solved by the following algorithm.

**Algorithm MWIS( $l$ )**

**Input:** An  $l$ claw-free graph  $G$ .

**Output:** A maximum weight independent set of  $G$ .

- (1) Execute Algorithm Gamma( $l$ ) for  $G$ . Let  $\mathcal{S}$  be the resulting family of subsets of  $V(G)$ .
- (2) For each  $H \in \mathcal{S}$ , compute a maximum weight independent set of  $G[H]$ . Then choose a best solution, i.e., one of maximum weight.

**Theorem 5** *Algorithm MWIS( $l$ ) is correct and can be executed in polynomial time.*

**Proof.** *Correctness:* By Lemma 4 (ii), Algorithm MWIS( $l$ ) is correct.

*Time bound:* By Lemma 5, step (1) can be executed in polynomial time. By Lemma 5, the family  $\mathcal{S}$  contains polynomially many members. Then by Lemma 4 (i) and by Theorem 1, step (2) can be executed in polynomial time. Thus, Algorithm MWIS( $l$ ) can be executed in polynomial time.  $\square$

Then one obtains the following result.

**Corollary 4** *For any fixed  $l$ , the MWIS problem can be solved in polynomial time for  $l$ claw-free graphs by Algorithm MWIS( $l$ ).*  $\square$

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